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Spin(9)-structures and connections with totally skew-symmetric torsion☆

Thomas Friedrich

Institut für Mathematik, Humboldt-Universität zu Berlin, Ziegelstrasse 13 A, Sitz: WBC Adlershof, D-10099 Berlin, Germany

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Abstract

We study Spin(9)-structures on 16-dimensional Riemannian manifolds and characterize the geometric types admitting a connection with totally skew-symmetric torsion.

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1. Introduction

The basic model in type II string theory is a 6-tuple $(M^n, g, \nabla, T, \Phi, \Psi)$ consisting of a Riemannian metric g, a metric connection ∇ with totally skew-symmetric torsion form T, a dilation function Φ and a spinor field Ψ. If the dilation function is constant, the string equations can be written in the following form (see [6,11,13,15,16]):

$$\mathrm{Ric}^{\nabla} = 0, \qquad \delta^g(T) = 0, \qquad \nabla\Psi = 0, \qquad T \cdot \Psi = 0.$$

Therefore, an interesting problem is the investigation of metric connections with totally skew-symmetric torsion. In [6] we proved that several non-integrable geometric structures (almost contact metric structures, almost complex structures, G_2-structures) admit a unique connection ∇ preserving it with totally skew-symmetric torsion. Moreover, we computed the corresponding torsion form T and we studied the integrability condition for ∇-parallel spinors as well as the Ricci tensor Ric^{∇}. In particular, we constructed seven-dimensional solutions of the string equations related to non-integrable G_2-structures. The five-dimensional

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E-mail address: friedric@mathematik.hu-berlin.de (T. Friedrich).

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case and its link with contact geometry was investigated in more details in the paper [7]. Similar results concerning eight-dimensional manifolds with a Spin(7)-structure are contained in the paper [12], the hyperKähler case was investigated in the papers [2,14,17]. Homogeneous models and the relation to Kostant's cubic Dirac operators were discussed in [1]. The aim of this note is to work out the case of 16-dimensional Riemannian manifolds with a non-integrable Spin(9)-structure. Gray [10] has pointed out that this special geometry may occur as a geometry with a weak holonomy group. Only recently we once again revisited the special Spin(9)-geometries in dimension 16 and, in particular, we proved that there are four basic classes (see [4]). Here we will study the problem which of these classes admit a connection ∇ with totally skew-symmetric torsion.

2. The geometry of Spin(9)-structures

The geometric types of Spin(9)-structures on 16-dimensional oriented Riemannian manifolds were investigated in the paper [4]. We summarize the basic facts defining this special geometry. Let us consider the 16-dimensional oriented Euclidean space $\mathbb{R}^{16}$. This space is the real spin representation of the group Spin(9) and, therefore, there exist nine linear operators $I_\alpha : \mathbb{R}^{16} \to \mathbb{R}^{16}$ such that the following relations hold:

$$I_\alpha^2 = \mathrm{Id}, \qquad I_\alpha^* = I_\alpha, \qquad I_\alpha \cdot I_\beta + I_\beta \cdot I_\alpha = 0 \quad (\alpha \neq \beta), \qquad \mathrm{Tr}(I_\alpha) = 0.$$

The subgroup $\mathrm{Spin}(9) \subset \mathrm{SO}(16)$ can be defined as the group of all automorphisms of $\mathbb{R}^{16}$ preserving, under conjugation, the nine-dimensional subspace $\mathbb{R}^9 := \mathrm{Lin}\{I_1, \dots, I_9\} \subset \mathrm{End}(\mathbb{R}^{16})$:

$$\mathrm{Spin}(9) := \{g \in \mathrm{SO}(\mathbb{R}^{16}) : g \cdot \mathbb{R}^9 \cdot g^{-1} = \mathbb{R}^9\}.$$

The decomposition of the Lie algebra $\mathfrak{so}(16) = \mathfrak{so}(9) \oplus \mathfrak{m}$ is explicitly given by

$$\mathfrak{so}(9) := \mathrm{Lin}\{I_\alpha \cdot I_\beta : \alpha < \beta\} = \Lambda^2(\mathbb{R}^9),$$
$$\mathfrak{m} := \mathrm{Lin}\{I_\alpha \cdot I_\beta \cdot I_\gamma : \alpha < \beta < \gamma\} = \Lambda^3(\mathbb{R}^9).$$

The operators $I_\alpha \cdot I_\beta$ and $I_\alpha \cdot I_\beta \cdot I_\gamma$ are skew-symmetric and, consequently, they define two systems of 2-forms $\omega_{\alpha\beta}$ and $\sigma_{\alpha\beta\gamma}$.

Let (M^{16}, g) be an oriented, 16-dimensional Riemannian manifold. A Spin(9)-structure is a nine-dimensional subbundle $V^9 \subset \mathrm{End}(TM^{16})$ of endomorphisms which is locally generated by sections I_α satisfying the algebraic relations described before. Denote by $\mathcal{F}(M^{16})$ the frame bundle of the oriented Riemannian manifold. Equivalently, a Spin(9)-structure is a reduction $\mathcal{R} \subset \mathcal{F}(M^{16})$ of the principal fiber bundle to the subgroup Spin(9). The Levi–Civita connection is a 1-form on $\mathcal{F}(M^{16})$ with values in the Lie algebra $\mathfrak{so}(16)$:

$$Z : T(\mathcal{F}(M^{16})) \to \mathfrak{so}(16).$$

We restrict the Levi–Civita connection to a fixed Spin(9)-structure $\mathcal{R}$ and decompose it with respect to the decomposition of the Lie algebra $\mathfrak{so}(16)$:

$$Z|_{T(\mathcal{R})} := Z^* \oplus \Gamma.$$

Then, Z^* is a connection in the principal Spin(9)-bundle $\mathcal{R}$ and Γ is a tensorial 1-form of type Ad, i.e., a 1-form on M^{16} with values in the associated bundle

$$\mathcal{R} \times_{\mathrm{Spin}(9)} \mathfrak{m} = \mathcal{R} \times_{\mathrm{Spin}(9)} \Lambda^3(\mathbb{R}^9) = \Lambda^3(V^9).$$

The Spin(9)-representation $\mathbb{R}^{16} \otimes \mathfrak{m} = \mathbb{R}^{16} \otimes \Lambda^3(\mathbb{R}^9)$ splits into four irreducible components:

$$\mathbb{R}^{16} \otimes \mathfrak{m} = \mathbb{R}^{16} \oplus \mathcal{P}_1(\mathbb{R}^9) \oplus \mathcal{P}_2(\mathbb{R}^9) \oplus \mathcal{P}_3(\mathbb{R}^9)$$

and, therefore, we obtain a similar decomposition of the bundle $\Lambda^1(M^{16}) \otimes \Lambda^3(V^9)$. The representation $\mathcal{P}_1(\mathbb{R}^9)$ has dimension 128. It is the restriction of the half spin representation Δ_{16}^- of Spin(16) to the subgroup Spin(9). The dimensions of the irreducible representations $\mathcal{P}_2(\mathbb{R}^9)$ and $\mathcal{P}_3(\mathbb{R}^9)$ are 432 and 768, respectively.

The decomposition of the section Γ yields the classification of all geometric types of Spin(9)-structures. In particular, there are four basic classes (see [4]). We remark that the sum $\mathcal{P}_1 \oplus \mathcal{P}_2$ is isomorphic to the bundle of 3-forms on M^{16}:

$$\Lambda^3(M^{16}) = \mathcal{P}_1(V^9) \oplus \mathcal{P}_2(V^9).$$

In order to fix the normalization, let us describe the embeddings $\Lambda^i(M^{16}) \to \Lambda^1(M^{16}) \otimes \Lambda^3(V^9)$, $i = 1, 3$, by explicit formulas. If $\mu^1 \in \Lambda^1(M^{16})$ is a (co-)vector, then the 1-form on M^{16} with values in the bundle $\Lambda^3(V^9)$ is given by

$$\mu^1 \mapsto \frac{1}{8} \sum_{\alpha<\beta<\gamma}^{9} I_\alpha I_\beta I_\gamma(\mu^1) \otimes I_\alpha \cdot I_\beta \cdot I_\gamma.$$

Similarly, if $\mu^3 \in \Lambda^3(M^{16})$ is a 3-form, we define

$$\mu^3 \mapsto \frac{1}{8} \sum_{\alpha<\beta<\gamma}^{9} (\sigma_{\alpha\beta\gamma} \lrcorner \mu^3) \otimes I_\alpha \cdot I_\beta \cdot I_\gamma,$$

where $\sigma_{\alpha\beta\gamma} \lrcorner \mu^3$ denotes the inner product of the 2-forms $\sigma_{\alpha\beta\gamma}$ by μ^3.

3. Spin(9)-connections with totally skew-symmetric torsion

We introduce the following equivariant maps:

$$\Phi : \mathbb{R}^{16} \otimes \mathfrak{spin}(9) \to \mathbb{R}^{16} \otimes S^2(\mathbb{R}^{16}),$$
$$\Phi(\Sigma)(X, Y, Z) := g(\Sigma(Z)(X), Y) + g(\Sigma(Y)(X), Z),$$
$$\Psi : \mathbb{R}^{16} \otimes \mathfrak{m} \to \mathbb{R}^{16} \otimes S^2(\mathbb{R}^{16}),$$
$$\Psi(\Gamma)(X, Y, Z) := g(\Gamma(Y)(X), Z) + g(\Gamma(Z)(X), Y).$$

It is well known (see [6]) that a geometric Spin(9)-structure admits a connection ∇ with totally skew-symmetric torsion if and only if $\Psi(\Gamma)$ is contained in the image of the homomorphism Φ. The representation $\mathbb{R}^{16} \otimes \mathfrak{spin}(9)$ splits into

$$\mathbb{R}^{16} \otimes \mathfrak{spin}(9) = \mathbb{R}^{16} \oplus \mathcal{P}_1(\mathbb{R}^9) \oplus \mathcal{P}_2(\mathbb{R}^9).$$

Consequently, if a Spin(9)-structure admits a connection ∇ with totally skew-symmetric torsion, then the $\mathcal{P}_3$-part of the form Γ must vanish. We split the Spin(9)-representation $\mathbb{R}^{16} \otimes S^2(\mathbb{R}^{16})$ into irreducible components. Since the symmetric linear maps I_α are traceless, the representation $\mathbb{R}^9$ is contained in $S^2_0(\mathbb{R}^{16})$ and we obtain the decomposition (see [4])

$$\mathbb{R}^{16} \otimes S^2(\mathbb{R}^{16}) = \mathbb{R}^{16} \oplus \mathbb{R}^{16} \otimes (\mathbb{R}^9 \oplus D^{126}) = 2 \cdot \mathbb{R}^{16} \oplus \mathcal{P}_1(\mathbb{R}^9) \oplus \mathbb{R}^{16} \otimes D^{126},$$

where $D^{126} := \Lambda^4(\mathbb{R}^9)$ is the unique irreducible representation of Spin(9) in dimension 126. Denote by D^{672} the unique irreducible Spin(9)-representation of dimension 672. Its highest weight is the 4-tuple $(3/2, 3/2, 3/2, 3/2)$.

Lemma 3.1. *The Spin(9)-representation $\mathbb{R}^{16} \otimes S^2(\mathbb{R}^{16})$ splits into the irreducible components*

$$\mathbb{R}^{16} \otimes S^2(\mathbb{R}^{16}) = 3 \cdot \mathbb{R}^{16} \oplus 2 \cdot \mathcal{P}_1(\mathbb{R}^9) \oplus \mathcal{P}_2(\mathbb{R}^9) \oplus \mathcal{P}_3(\mathbb{R}^9) \oplus D^{672}.$$

Proof. Since $\mathbb{R}^{16} \otimes \mathfrak{m}$ contains the representations $\mathcal{P}_2(\mathbb{R}^9)$, $\mathcal{P}_3(\mathbb{R}^9)$ and Ψ is nontrivial, the tensor product $\mathbb{R}^{16} \otimes D^{126}$ contains the two representations, too. Moreover, the highest weights of $\mathbb{R}^{16}$ and D^{126} are $(1/2, 1/2, 1/2, 1/2)$ and $(1, 1, 1, 1)$, respectively. Then the tensor product $\mathbb{R}^{16} \otimes D^{126}$ contains the representation D^{672} of the highest weight $(3/2, 3/2, 3/2, 3/2)$ (see [9, p. 425]). Consequently, we obtain

$$\mathbb{R}^{16} \otimes D^{126} = \mathcal{P}_2(\mathbb{R}^9) \oplus \mathcal{P}_3(\mathbb{R}^9) \oplus D^{672} \oplus S,$$

where the dimension of the rest equals $\dim(S) = 144$. The representation S is not an SO(9)-representation. The list of small-dimensional Spin(9)-representations yields that $S = \mathbb{R}^{16} \oplus \mathcal{P}_1(\mathbb{R}^9)$, the final result. The decomposition of $\mathbb{R}^{16} \otimes D^{126}$ can be computed by a suitable computer program, too. □

Lemma 3.2. *For any two vectors $X, Y \in \mathbb{R}^{16}$ the following identity holds*:

$$\sum_{\alpha<\beta}^{9} \omega_{\alpha\beta}(X, Y) \cdot \omega_{\alpha\beta} + \sum_{\alpha<\beta<\gamma}^{9} \sigma_{\alpha\beta\gamma}(X, Y) \cdot \sigma_{\alpha\beta\gamma} = 8 \cdot X \wedge Y.$$

Proof. The 2-forms $\omega_{\alpha\beta}$ and $\sigma_{\alpha\beta\gamma}$ constitute a basis of the space $\Lambda^2(\mathbb{R}^{16})$ of all 2-forms in 16 variables. Therefore, the identity is simply the decomposition of the 2-form $X \wedge Y$ with respect to this basis. Remark that the length of the basic forms $\omega_{\alpha\beta}$ and $\sigma_{\alpha\beta\gamma}$ equals $2 \cdot \sqrt{2}$. □

Theorem 3.1. *A Spin(9)-structure on a 16-dimensional Riemannian manifold M^{16} admits a connection ∇ with totally skew-symmetric torsion if and only if the $(\mathbb{R}^{16} \oplus \mathcal{P}_3)$-part of the form Γ vanishes. In this case Γ is a usual 3-form on the manifold M^{16}, the connection ∇ is unique and its torsion form T is given by the formula $T = -2 \cdot \Gamma$.*

Proof. For a fixed vector $\Gamma \in \mathbb{R}^{16}$ the tensor $\Psi(\Gamma)(X, Y, Y)$ is given by the formula

$$\Psi(\Gamma)(X, Y, Y) = \frac{1}{4} \sum_{\alpha<\beta<\gamma}^{9} \sigma_{\alpha\beta\gamma}(\Gamma, Y) \cdot \sigma_{\alpha\beta\gamma}(X, Y).$$

Since the multiplicity of $\mathbb{R}^{16}$ in the representation $\mathbb{R}^{16} \otimes \mathfrak{spin}(9)$ equals 1, any Spin(9)-equivariant map $\Sigma : \mathbb{R}^{16} \to \mathbb{R}^{16} \otimes \mathfrak{spin}(9)$ is a multiple of

$$\Sigma(\Gamma) = \sum_{\alpha<\beta}^{9} I_{\alpha\beta}(\Gamma) \otimes I_{\alpha\beta}.$$

Consequently, if $\Psi(\Gamma)$ is in the image of Φ, there exists a constant c such that

$$\sum_{\alpha<\beta<\gamma}^{9} \sigma_{\alpha\beta\gamma}(\Gamma, Y) \cdot \sigma_{\alpha\beta\gamma}(X, Y) = c \cdot \sum_{\alpha<\beta}^{9} \omega_{\alpha\beta}(\Gamma, Y) \cdot \omega_{\alpha\beta}(X, Y).$$

For $\Gamma = X = e_{16}$ we compute the corresponding quadratic forms in the variables $y_1, \dots, y_{16}$:

$$\Psi(e_{16}) = \sum_{i=1}^{8} y_i^2 + 4 \cdot \sum_{j=9}^{15} y_j^2, \qquad \Phi(\Sigma(e_{16})) = 7 \cdot \sum_{i=1}^{8} y_i^2 + 4 \cdot \sum_{j=9}^{15} y_j^2$$

a contradiction. Next consider the case that $\Gamma \in \Lambda^3(\mathbb{R}^{16})$ is a 3-form. By Lemma 3.2 we have

$$\begin{aligned}\Psi(\Gamma)(X, Y, Y) &= \frac{1}{4} \sum_{\alpha\beta\gamma}^{9} \Gamma(\sigma_{\alpha\beta\gamma}, Y) \cdot \sigma_{\alpha\beta\gamma}(X, Y) \\ &= -\frac{1}{4} \sum_{\alpha\beta}^{9} \Gamma(\omega_{\alpha\beta}, Y) \cdot \omega_{\alpha\beta}(X, Y) + 2 \cdot \Gamma(X, Y, Y).\end{aligned}$$

Since Γ is a 3-form, the term $\Gamma(X, Y, Y)$ vanishes. Let us introduce

$$\Sigma(\Gamma) := -\frac{1}{8} \sum_{\alpha\beta}^{9} (\omega_{\alpha\beta} \lrcorner \Gamma) \otimes \omega_{\alpha\beta}.$$

Then $\Sigma(\Gamma)$ belongs to the space $\mathbb{R}^{16} \otimes \mathfrak{spin}(9)$ and we have $\Phi(\Sigma(\Gamma)) = \Psi(\Gamma)$. Consequently, in case Γ is a 3-form on M^{16}, there exists a unique connection ∇ preserving the Spin(9)-structure with totally skew-symmetric torsion. Its torsion form T is basically given by the difference $\Gamma(X) - \Sigma(\Gamma)(X)$ (see [6]) and we obtain the formula $T = -2 \cdot \Gamma$. □

Let us characterize Spin(9)-structures of type $\mathcal{P}_1 \oplus \mathcal{P}_2$ using the Riemannian covariant derivatives ∇I_α of the symmetric endomorphisms describing the structure. For an arbitrary 2-form S we introduce the symmetric forms by the formula

$$S_\alpha(Y, Z) := -S(I_\alpha(Y), Z) + S(Y, I_\alpha(Z)), \quad \alpha = 1, \dots, 9.$$

The connection ∇ preserves the nine-dimensional bundle of endomorphisms I_α and therefore there exist 1-forms $M_{\alpha\beta}$ such that $\nabla I_\alpha = \sum_{\beta=1}^{9} M_{\alpha\beta} \cdot I_\beta$. Since $\nabla_X Y = \nabla^g_X + 1/2 \cdot T(X, Y, \cdot)$ we obtain the following formula for the Riemannian covariant derivative of the endomorphisms I_α

$$\nabla^g_X I_\alpha = \sum_{\beta=1}^{9} M_{\alpha\beta}(X) \cdot I_\beta + \frac{1}{2} \cdot (X \lrcorner T)_\alpha,$$

where T is a 3-form. The latter equation characterizes Spin(9)-structures of type $\mathcal{P}_1 \oplus \mathcal{P}_2$.

4. Homogeneous Spin(9)-structures

Consider a Lie group G, a subgroup H and suppose that the homogeneous space G/H is naturally reductive of dimension 16. We fix a decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}, \quad [\mathfrak{h}, \mathfrak{n}] \subset \mathfrak{n}, \quad \mathfrak{n} = \mathbb{R}^{16}$$

as well as a scalar product $(,)_\mathfrak{n}$ such that for all $X, Y, Z \in \mathfrak{n}$

$$([X, Y]_\mathfrak{n}, Z)_\mathfrak{n} + (Y, [X, Z]_\mathfrak{n})_\mathfrak{n} = 0$$

holds, where $[X, Y]_\mathfrak{n}$ denotes the $\mathfrak{n}$-part of the commutator. Moreover, suppose that the isotropy representation leaves a Spin(9)-structure in the vector space $\mathfrak{n}$ invariant. Then G/H admits a homogeneous Spin(9)-structure. Indeed, the frame bundle is an associated bundle:

$$\mathcal{F}(G/H) = G \times_{\mathrm{Ad}} \mathrm{SO}(\mathfrak{n})$$

and $\mathcal{R} := G \to \mathcal{F}(G/H)$ is a reduction to the subgroup H contained in Spin(9). The canonical connection ∇^{can} of the reductive space preserves the Spin(9)-structure and has totally skew-symmetric torsion:

$$T^{\nabla^{\mathrm{can}}}(X, Y, Z) = -([X, Y]_\mathfrak{n}, Z)_\mathfrak{n}.$$

Consequently, any homogeneous Spin(9)-structure admits an affine connection with totally skew-symmetric torsion, i.e., it is of type $\mathcal{P}_1 \oplus \mathcal{P}_2$.

Corollary 4.1. *Any homogeneous Spin*(9)*-structure on a naturally reductive space* $M^{16} = G/H$ *is of type* $\mathcal{P}_1 \oplus \mathcal{P}_2$.

Remark 4.1. In particular, for any homogeneous Spin(9)-structure the difference Γ between the Levi–Civita connection and the canonical connection is a 3-form. Indeed, the Levi–Civita connection of a reductive space is given by the map $\mathfrak{n} \to \mathrm{End}(\mathfrak{n})$

$$X \mapsto \tfrac{1}{2} \cdot [X, \cdot]_\mathfrak{n}.$$

Then we obtain

$$\Gamma(X) = \frac{1}{2} \cdot \mathrm{pr}_{\mathfrak{m}}([X, \cdot]_{\mathfrak{n}}) = \frac{1}{32} \sum_{i,j=1}^{16} \sum_{\alpha<\beta<\gamma} ([X, e_i]_{\mathfrak{n}}, e_j)_{\mathfrak{n}} \cdot \sigma_{\alpha\beta\gamma}(e_i, e_j) \cdot \sigma_{\alpha\beta\gamma}.$$

We write the latter equation in the following form

$$\Gamma(X) = -\frac{1}{16} \sum_{\alpha<\beta<\gamma} (\sigma_{\alpha\beta\gamma} \lrcorner T^{\nabla^{\mathrm{can}}})(X) \cdot \sigma_{\alpha\beta\gamma} = -\frac{1}{2} \cdot T^{\nabla^{\mathrm{can}}}(X, \cdot, \cdot),$$

i.e., Γ is proportional to the torsion of the canonical connection:

$$\Gamma(X)(Y, Z) = -\tfrac{1}{2} \cdot T^{\nabla^{\mathrm{can}}}(X, Y, Z).$$

There are homogeneous Spin(9)-structures on different reductive spaces (see [4]).

Example 4.1. The group Spin(9) acts transitively on the sphere S^{15}, the isotropy group is isomorphic to Spin(7) and the isotropic representation of the reductive space $S^1 \times S^{15} = (S^1 \times \mathrm{Spin}(9))/\mathrm{Spin}(7)$ is contained in Spin(9).

Example 4.3. The space $S^1 \times S^1 \times (\mathrm{SO}(8)/\mathrm{G}_2)$ admits a homogeneous Spin(9)-structure.

Example 4.3. The space $\mathrm{SU}(5)/\mathrm{SU}(3)$ admits a homogeneous Spin(9)-structure.

5. G-connections with totally skew-symmetric torsion

The class of Spin(9)-structures corresponding to the representation $\mathbb{R}^{16} \subset \mathbb{R}^{16} \otimes \mathfrak{m}$ is related with conformal changes of the metric. Indeed, if (M^{16}, g, V^9) is a Riemannian manifold with a fixed Spin(9)-structure $V^9 \subset \mathrm{End}(TM^{16})$ and $g^* = \mathrm{e}^{2f} \cdot g$ is a conformal change of the metric, then the triple (M^{16}, g^*, V^9) is a Riemannian manifold with a Spin(9)-structure, too. The fact that the 16-dimensional class of Spin(9)-structures corresponding to $\mathbb{R}^{16}$ is not admissible in Theorem 3.1 means that the existence of a connection with totally skew-symmetric torsion and preserving a Spin(9)-structure is not invariant under conformal transformations of the metric. From this point of view the behavior of Spin(9)-structures is different from the behavior of G_2-structures, Spin(7)-structures, quaternionic Kähler structures or contact structures (see [7,8,12,14]). We will explain this effect in a more general context.

Let $\mathrm{G} \subset \mathrm{SO}(n)$ be a closed subgroup of the orthogonal group and decompose the Lie algebra

$$\mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{m}.$$

A G-structure of a Riemannian manifolds M^n is a reduction $\mathcal{R} \subset \mathcal{F}(M^n)$ of the frame bundle to the subgroup G. The Levi–Civita connection is a 1-form Z on $\mathcal{F}(M^n)$ with values in the Lie algebra $\mathfrak{so}(n)$. We restrict the Levi–Civita connection to a fixed G-structure $\mathcal{R}$ and decompose it with respect to the decomposition of the Lie algebra $\mathfrak{so}(n)$:

$$Z|_{T(\mathcal{R})} := Z^* \oplus \Gamma.$$

Then, Z^* is a connection in the principal G-bundle $\mathcal{R}$ and Γ is a tensorial 1-form of type Ad, i.e., a 1-form on M^n with values in the associated bundle $\mathcal{R} \times_{\mathrm{G}} \mathfrak{m}$. The G-representation $\mathbb{R}^n \otimes \mathfrak{m}$ splits into irreducible components and the corresponding decomposition of Γ characterizes the different non-integrable G-structures. We introduce the equivariant maps:

$$\Phi : \mathbb{R}^n \otimes \mathfrak{g} \to \mathbb{R}^n \otimes S^2(\mathbb{R}^n), \quad \Phi(\Sigma)(X, Y, Z) := g(\Sigma(Z)(X), Y) + g(\Sigma(Y)(X), Z),$$
$$\Psi : \mathbb{R}^n \otimes \mathfrak{m} \to \mathbb{R}^n \otimes S^2(\mathbb{R}^n), \quad \Psi(\Gamma)(X, Y, Z) := g(\Gamma(Y)(X), Z) + g(\Gamma(Z)(X), Y).$$

It is well known (see [6]) that a geometric G-structure admits a connection ∇ with totally skew-symmetric torsion if and only if $\Psi(\Gamma)$ is contained in the image of the homomorphism Φ. There is an equivalent formulation of this condition. Indeed, let us introduce the maps

$$\Theta_1 : \Lambda^3(\mathbb{R}^n) \to \mathbb{R}^n \otimes \mathfrak{m}, \qquad \Theta_2 : \Lambda^3(\mathbb{R}^n) \to \mathbb{R}^n \otimes \mathfrak{g}$$

given by the formulas

$$\Theta_1(T) := \sum_i (\sigma_i \lrcorner T) \otimes \sigma_i, \qquad \Theta_2(T) := \sum_j (\mu_j \lrcorner T) \otimes \mu_j,$$

where σ_i is an orthonormal basis in $\mathfrak{m}$ and μ_j is an orthonormal basis in $\mathfrak{g}$. Observe that the kernel of the map $(\Psi \oplus \Phi) : \mathbb{R}^n \otimes \mathfrak{so}(n) \to \mathbb{R}^n \otimes S^2(\mathbb{R}^n)$ coincides with the image of the map $(\Theta_1 \oplus \Theta_2) : \Lambda^3(\mathbb{R}^n) \to \mathbb{R}^n \otimes \mathfrak{so}(n)$. Consequently, for any element $\Gamma \in \mathbb{R}^n \otimes \mathfrak{m}$, the condition $\Psi(\Gamma) \in \mathrm{Image}(\Phi)$ is equivalent to $\Gamma \in \mathrm{Image}(\Theta_1)$.

Theorem 5.1. *A G-structure $\mathcal{R} \subset \mathcal{F}(M^n)$ of a Riemannian manifold admits a connection ∇ with totally skew-symmetric torsion if and only if the* 1*-form Γ belongs to the image of Θ_1, $\Gamma = \Theta_1(T)$. In this case the* 3*-form $(-2 \cdot T)$ is the torsion form of the connection.*

Consequently, only such geometric types (i.e., irreducible components of $\mathbb{R}^n \otimes \mathfrak{m}$) are admissible which occur in the G-decomposition of $\Lambda^3(\mathbb{R}^n)$. This explains the different behavior of G-structures with respect to conformal transformations.

Example 5.1. In case of $\mathrm{G} = \mathrm{Spin}(9)$ we have

$$\mathbb{R}^{16} \otimes \mathfrak{m} = \mathbb{R}^{16} \oplus \Lambda^3(\mathbb{R}^{16}) \oplus \mathcal{P}_3(\mathbb{R}^9)$$

and the $\mathbb{R}^{16}$-component is not contained in $\Lambda^3(\mathbb{R}^{16}) = \mathcal{P}_1(\mathbb{R}^9) \oplus \mathcal{P}_2(\mathbb{R}^9)$, i.e., a conformal change of a Spin(9)-structure does not preserve the property that the structure admits a connection with totally skew-symmetric torsion.

Example 5.2. In case of a seven-dimensional G_2-structure the situation is different. Indeed, we decompose the G_2-representation (see [6])

$$\Lambda^3(\mathbb{R}^7) = \mathbb{R}^1 \oplus \mathbb{R}^7 \oplus \Lambda^3_{27}, \qquad \mathbb{R}^7 \otimes \mathfrak{m} = \mathbb{R}^1 \oplus \mathbb{R}^7 \oplus \Lambda^2_{14} \oplus \Lambda^3_{27}$$

and, consequently, a conformal change of a G_2-structure preserves the property that the structure admits a connection with totally skew-symmetric torsion.

Example 5.3. Let us consider Spin(7)-structures on eight-dimensional Riemannian manifolds. The subgroup $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$ is the real Spin(7)-representation $\Delta_7 = \mathbb{R}^8$. The complement $\mathfrak{m} = \mathbb{R}^7$ is the standard seven-dimensional representation and the Spin(7)-structures on an eight-dimensional Riemannian manifold M^8 correspond to the irreducible components of the tensor product

$$\mathbb{R}^8 \otimes \mathfrak{m} = \mathbb{R}^8 \otimes \mathbb{R}^7 = \Delta_7 \otimes \mathbb{R}^7 = \Delta_7 \oplus K,$$

where K denotes the kernel of the Clifford multiplication $\Delta_7 \otimes \mathbb{R}^7 \to \Delta_7$. It is well known that K is an irreducible Spin-representation. Therefore, there are only two basic types of Spin(7)-structures (see [3]). On the other hand, the map $\Lambda^3(\mathbb{R}^8) \to \mathbb{R}^8 \otimes \mathfrak{m}$ is injective and the Spin(7)-representation $\Lambda^3(\mathbb{R}^8) = \Lambda^3(\Delta_7)$ splits again into the irreducible components

$$\Lambda^3(\Delta_7) = \Delta_7 \oplus K,$$

i.e., $\Lambda^3(\mathbb{R}^8) \to \mathbb{R}^8 \otimes \mathfrak{m}$ is an isomorphism. Theorem 5.1 yields immediately that any Spin(7)-structure on an eight-dimensional Riemannian manifold admits a connection with totally skew-symmetric torsion (see [12]). We remark that $n = 8$ is the smallest dimension where this effect can occur. Indeed, let $\mathrm{G} \subset \mathrm{SO}(n)$ be a subgroup of dimension g and suppose that any G-structure admits a connection with totally skew-symmetric torsion, i.e., the map $\Lambda^3(\mathbb{R}^n) \to \mathbb{R}^n \otimes \mathfrak{m}$ is surjective. On the other side, the isotropy representation $\mathrm{G} \to \mathrm{SO}(\mathfrak{m})$ of the compact Riemannian manifold $\mathrm{SO}(n)/\mathrm{G}$ is injective. Consequently, we obtain the inequalities

$$\frac{1}{3}(n^2 - 1) \leq \mathrm{g} \leq \frac{1}{2}(n^2 - 3n + 2).$$

The minimal pair satisfying this condition is $n = 8$, $\mathrm{g} = 21$. Using not only the dimension of the G-representation one can exclude other dimensions, for example $n = 9$. For a further discussion see (see [5]).

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